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Chromatic Ramsey Theory

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Let $\mathfrak{G}$ be a countable graph which has infinite chromatic number. If γ is a coloring of $[G]^2$ with two colors, is there then a subset $H \subseteq G$ such that γ is constant on $[H]^2$ and $\mathfrak{G}|_H$, the graph induced by $\mathfrak{G}$ on H, has infinite chromatic number? As edges and non-edges can be colored with different colors this will be the case iff $\mathfrak{G}$ contains an infinite clique. It turns out that if the clique size of $\mathfrak{G}$ is unbounded but $\mathfrak{G}$ does not contain an infinite clique then for every coloring of $[G]^2$ with τ colors, there are some two of the τ colors such that there is an infinite chromatic subgraph of $\mathfrak{G}$ the vertex set of which forms only pairs colored in those two colors; and this is best possible, because one can always distinguish between edges and non-edges. In the case in which the graphs do not contain the complete graph on n vertices the situation is much more complicated. We will show that for every $3 \leqslant n \in \omega$ there is a graph $\mathfrak{G}$, which does not embed the complete graph on n vertices, with the property that for every positive number τ there exists a coloring of $[G]^2$ with τ colors such that the vertex set of every infinite chromatic subgraph of $\mathfrak{G}$ forms pairs in each of the τ colors. On the other hand there is a graph $\mathfrak{G}$, which does not embed the complete graph on n vertices, and which has the property that for every positive number τ and every coloring of $[G]^2$ with τ colors there is an infinite chromatic subgraph of $\mathfrak{G}$ the pairs of which use at most 3 colors. We will generalize to the case of colorings of k-element subsets.

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1. Introduction

We consider a graph $\mathfrak{G}$ to be a set of two-element sets. Hence our graphs do not contain multiple edges or loops. We will usually denote the set of vertices $\bigcup \mathfrak{G}$ of $\mathfrak{G}$ by G. If H is a subset of G, then the *induced subgraph* $\mathfrak{G}|_H$ of $\mathfrak{G}$ is the graph $\mathfrak{H} = \mathfrak{G} \cap [H]^2$; where, as usual, $[H]^\tau$ denotes the set of τ-element subsets of H. The family of graphs which do not contain a complete graph on n vertices will be denoted by $\mathcal{N}_n$. For cardinals κ, ρ, σ and ν we write

$$\mathfrak{G} \leftrightarrow_\sigma (\kappa)^\nu_\rho$$

to mean that for every coloring $\gamma : [G]^\nu \to \rho$ there exists a subset $S \subseteq \rho$ with $|S|\sigma$, a subset $H \subseteq G$ such that $\gamma([H]^\nu) \subseteq S$ and the chromatic number of the graph $\mathfrak{G}|_H$ is at least κ. The negation of $\mathfrak{G} \leftrightarrow_\sigma (\kappa)^\nu_\rho$ is $\mathfrak{G} \not\leftrightarrow_\sigma (\kappa)^\nu_\rho$. In this paper we are interested in the relation $\mathfrak{G} \leftrightarrow_s (\mathcal{N}_0)^k_\tau$ for finite cardinals τ, s and k. For $c \geqslant d$ two positive numbers, we obtain

$$\mathfrak{G} \leftrightarrow_s (\mathcal{N}_0)^k_\tau \quad \text{implies} \quad \mathfrak{G} \leftrightarrow_{s+c} (\mathcal{N}_0)^k_{\tau+d}. \tag{1}$$

and

$$\mathfrak{G} \leftrightarrow_s (\mathcal{N}_0)^k_{\tau+d} \quad \text{implies} \quad \mathfrak{G} \leftrightarrow_s (\mathcal{N}_0)^k_\tau. \tag{2}$$

To see (1), just combine $d+1$ colors of a coloring with $\tau + d$ colors into one color to obtain a coloring with τ colors. Equation (2) follows from the observation that every coloring with τ colors can be interpreted to be a coloring with $\tau + d$ colors. If $\mathfrak{G}$ contains an infinite complete subgraph $\mathfrak{H}$ and γ is a coloring of $[G]^k$ with τ colors, then apply the ordinary Ramsey Theorem [9] to the restriction of γ to $[H]^k$, to deduce that H contains an infinite subset S such that γ is the constant function on $[S]^k$. Hence we obtain the easy Theorem 1,

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THEOREM 1. *If* $\mathfrak{G}$ *contains an infinite clique and* $1 \leq k$, $\tau \in \omega$, *then* $\mathfrak{G} \leftrightarrow_1 (\mathcal{N}_0)^k_r$.

We will denote by $p(k) = 2^{k-1}$ the number of ordered partitions of the number k. That is, $p(k)$ is equal to the number of sequences of positive integers of the form $(\tau_1, \tau_2, \ldots, \tau_w)$ such that $\tau_1 + \tau_2 + \cdots + \tau_w = k$. We will prove the following.

THEOREM 2. *If, for every* $n \in \omega$, $\mathfrak{G} \notin \mathcal{N}_n$, *then, for every* $\tau \in \omega$, $\mathfrak{G} \leftrightarrow_{p(k)} (\aleph_0)^k_\tau$.

THEOREM 3. *If, for every* $n \in \omega$, $\mathfrak{G} \notin \mathcal{N}_n$ *but* $\mathfrak{G}$ *does not contain an infinite complete subgraph, then* $\mathfrak{G} \not\leftrightarrow_{p(k)-1} (\aleph_0)^k_{p(k)}$.

Note that the above three theorems together with equations (1) and (2) provide a complete answer for graphs with unbounded clique sizes. The situation seems to be much more complicated in the case of the families $\mathcal{N}_n$ of graphs, as the following two theorems indicate.

THEOREM 4. *There exists a triangle-free graph* $\mathfrak{G}$ *such that, for every* $1 \leq \tau \in \omega$, $\mathfrak{G} \leftrightarrow_3 (\aleph_0)^2_\tau$.

THEOREM 5. *There exists a triangle-free graph* $\mathfrak{G}$ *such that, for every* $2 \leq \tau \in \omega$, $\mathfrak{G} \not\leftrightarrow_{\tau-1} (\aleph_0)^2_\tau$.

Let $\mathfrak{U}_n$ be the homogeneous graph which embeds every finite member of $\mathcal{N}_n$. It is well known [5] that every countable graph in $\mathcal{N}_n$ can be embedded into $\mathfrak{U}_n$. Hence it follows from Theorem 4 that for $3 \leq n$ and $1 \leq \tau \in \omega$, $\mathfrak{U}_n \leftrightarrow_3 (\aleph_0)^2_\tau$. This is best possible, since we can show the following.

THEOREM 6. *For* $3 \leq n \in \omega$ *and* $3 \leq \tau \in \omega$, $\mathfrak{U}_n \not\leftrightarrow_2 (\aleph_0)^2_r$.

The reader will note that in the case in which clique sizes are bounded, we do not do as well as the partition number, since $p(2) = 2 < 3$.

Let $(C_i; i \in \omega)$ be a sequence of pairwise disjoint finite sets such that, for all $i \in \omega$, $(|C_i| < |C_{i+1}|)$. A *stepgraph* $\mathfrak{C}$ *with partition* $C_0, C_1, C_2, \ldots$ is a graph such that $C = \bigcup_{i \in \omega} C_i$, there are no edges between different C_i's and different C_i's are pairwise disjoint. The stepgraph is said to be *connected* if each C_i is a connected component of $\mathfrak{C}$. The stepgraph $\mathfrak{C}$ is *complete* if each of the C_i induces a clique of $\mathfrak{C}$. For $i \in \omega$ we will put $P_i(\mathbb{C}) = C_i$ A refinement of the stepgraph $\mathfrak{C}$ is a subgraph of $\mathfrak{C}$ which is itself a stepgraph. For a finite subset K of C, the number $i \in \omega$ is *significant* if $K \cap C_i \neq \emptyset$. Let $i_1 < i_2 < \cdots < i_w$ be the significant numbers for K. Then the ordered partition $(|K \cap C_{i_1}|, |K \cap C_{i_2}|, \ldots, |K \cap C_{i_w}|)$ of $|K|$ is the *partition type* of K, and the partition of K *induced by* $\mathfrak{C}$ is $P_1(K) = K \cap C_{i_1}$, $P_2(K) = K \cap C_{i_2}, \ldots, P_w(K) = K \cap C_{i_w}$. If $\tau = (s_1, s_2, \ldots, s_w)$ is an ordered partition of the number $k \geq 1$, we will denote by $\mathcal{T}_\tau(\mathfrak{C})$ the set of all subsets of C the partition type of which is τ. If $\gamma: \mathcal{T}_\tau(\mathfrak{C}) \to \tau$ is a coloring of the elements of $\mathcal{T}_\tau(\mathfrak{C})$ with τ colors, then we will say the following.

The coloring γ is *uniform* if any two subsets K and L in $\mathcal{T}_\tau(\mathfrak{C})$ with $P_i(K) = P_i(L)$ for all $1 \leq i \leq w - 1$ have the same color.

The coloring γ is *uniform up to n*, for $n \in \omega$, if any two subsets K and L in $\mathcal{T}_\tau(\mathfrak{C})$ for which $P_w(K) \cup P_w(L) \subseteq \bigcup_{i \geq n} C_i$ and which have the property that $P_i(K) = P_i(L) \subseteq \bigcup_{i<n} C_i$ for all $1 \leq i \leq w - 1$, have the same color.

For τ, k, $t \geq 1$, we will denote by $R(\tau, k, t)$ the ordinary Ramsey number. That is, $R(\tau, k, t)$ is the smallest number such that for every set S with $|S| \geq R(\tau, k, t)$ and every coloring $\gamma : [S]^k \to \tau$, there is a subset $T \subseteq S$ with $|T| \geqslant t$ such that γ is constant on $[T]^k$ (see [9]).

2. The Results for Graphs with Unbounded Clique Sizes

Lemma 1. *Let $\mathfrak{C}$ be a stepgraph with partition $(C_0, C_1, C_2, \ldots)$, $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$ an ordered partition of the number k and $\gamma : \mathcal{T}_\tau(\mathbb{C}) \to \tau$ a coloring of the k-element subsets of C, which have partition type τ with $\tau \geqslant 1$ colors. Then there is, for every $n \in \omega$, a refinement $\mathfrak{D}$ of $\mathfrak{C}$ with $P_i(\mathfrak{D}) = C_i$ for all $i < n$ and such that the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform up to n. Furthermore, if $w = 1$ and $n = 0$, then the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathscr{D})$ is the constant function.*

Proof. For $w \geqslant 2$, let the ordered partition σ of $k - \tau_w$ be given by $\sigma = (\tau_1, \tau_2, \ldots, \tau_{w-1})$ and the set S by

$$S = \mathcal{T}_\sigma(\mathfrak{C}) \cap \left[\bigcup_{i<n} C_i\right]^{k-\tau_w}.$$

We define the number c to be

$$c = \begin{cases} \tau, & \text{if } w = 1; \\ |\tau^S|, & \text{otherwise.} \end{cases}$$

Note that S is the set of all $(k - \tau_w)$-element subsets of $\bigcup_{i<n} C_i$ which have partition type σ. Hence $|\tau^S|$ is the number of colorings with τ colors of the set of all $(k - \tau_w)$-element subsets of $\bigcup_{i<n} C_i$, which have partition type σ. Let $j \geqslant n$ and $T \subseteq C_j$, with $|T| = \tau_w$. Note that if $S \in S$ then $S \cup T \in \mathcal{T}_\tau(\mathfrak{C})$. We can therefore define a coloring γ_T of S, by putting $\gamma_T(S) = \gamma(S \cup T)$ for $S \in S$. Then $\gamma_T \in [S]^\tau$. The association of T with γ_T is a coloring Γ of $\bigcup_{i \geqslant n} [C_i]^{r_w}$ with $\|[r^S]\| = c$ colors.

Choose a strictly increasing sequence of numbers $(s_i; i \in \omega)$ such that, for $i < n$, $s_i = |C_i|$. There exists then a sequence t_i such that, for $i < n$, $t_i = i$ and, for all $i \geqslant n$, $|C_{t_i}| \geqslant R(c, w, s_i)$. For $i \geqslant n$, using the definition of the number $R(c, w, s_i)$, let B_i be a subset, $B_i \subseteq C_{t_i}$, such that Γ is constant on $[B_i]^w$. Put $B_i = C_i$ for $i < n$ and $\mathfrak{B} = \mathfrak{C}|_{\bigcup_{i \in \omega} B_i}$. Because the coloring Γ is finite, there is a refinement $\mathfrak{D}$ of $\mathfrak{B}$ such that Γ is constant on $\bigcup_{i \geqslant n} [P_i(\mathfrak{D})]^{\tau_w}$. Clearly, $\mathfrak{D}$ is the desired refinement of $\mathfrak{C}$ for which the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform up to n. □

Lemma 2. *Let $\mathfrak{C}$ be a stepgraph, τ an ordered partition of the number k and $\gamma : \mathcal{T}_\tau(\mathfrak{C}) \to \tau$ a coloring of the k-element subsets of C which have partition type τ with $\tau \geqslant 1$ colors. Then there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ such that the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform.*

Proof. Let $\mathfrak{C}_0 = \mathfrak{C}$ and, for $0 < n \in \omega$, let $\mathfrak{C}_n$ be a refinement of $\mathfrak{C}_{n-1}$ such that the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{C}_n)$ is uniform up to n, and such that, for all $i < n$, $P_i(\mathfrak{C}_n) = P_i(\mathfrak{C}_{n-1})$. Note that, for all $n, m \in \omega$ and $i \leqslant n$, $P_i(\mathfrak{C}_n) = P_i(\mathfrak{C}_{n+m})$. Therefore,

$$D = \bigcap_{i \in \omega} C_i \neq \varnothing,$$

and $\mathfrak{D} = \mathfrak{C}|_D$ is a stepgraph, and hence a refinement of $\mathfrak{C}$. Also, of course, the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform. □

Lemma 3. *Let $\mathfrak{C}$ be a stepgraph, $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$ an ordered partition of the* *number k and $\gamma:\mathcal{T}_\tau(\mathfrak{C})\to\tau$ a coloring of the k-element subsets of C, which have partition type τ, with $\tau\geqslant 1$ colors. Then there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ such that the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is the constant function.*

Proof. We proceed by induction on w. For $w=1$ we use Lemma 1 with $n=0$. It follows from Lemma 2 that we may assume that the coloring γ is uniform on $\mathcal{T}_\tau(\mathfrak{C})$. Let σ be the ordered partition of $k-\tau_w$ given by $\sigma=(\tau_1,\tau_2,\ldots,\tau_{w-1})$. Note, then, that if $S\in\mathcal{T}_\sigma(\mathfrak{C})$ and R and T are any two subsets of C such that $S\cup R\in\mathcal{T}_\tau(\mathfrak{C})$ and $S\cup T\in\mathcal{T}_\tau(\mathfrak{C})$, then $\gamma(S\cup R)=\gamma(S\cup T)$. Hence we can associate with any such $S\in\mathcal{T}_\sigma(\mathfrak{C})$ the color $\gamma^*(S)=\gamma(S\cup T)$ for any T such that $S\cup T\in\mathcal{T}_\tau(\mathfrak{C})$. Using induction, let $\mathfrak{D}$ be a refinement of $\mathfrak{C}$ such that γ^* is constant. Clearly, then, the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is constant. □

Lemma 4. *If $\mathfrak{C}$ is a complete stepgraph and $0<k\in\omega$, then, for all $1\leqslant\tau\in\omega$,*

$$\mathfrak{C}\leftrightarrow_{p(k)}(\aleph_0)^k_\tau.$$

Proof. The proof follows easily from Lemma 3. □

Lemma 5. *Every graph $\mathfrak{G}=(G,E)$ in which the clique sizes are unbounded but which does not contain an infinite complete subgraph contains a complete stepgraph as an induced subgraph.*

Proof. There exists a sequence of pairwise disjoint subsets $(C_i; i\in\omega)$ such that the sequence $(|C_i|; i\in\omega)$ is strictly increasing, and for each $i\in\omega$ the graph $\mathfrak{G}|_{C_i}$ is complete. This follows from the fact that every finite such sequence can be extended.

Let $\mathfrak{C}$ be the stepgraph such that $P_i(\mathfrak{C})=C_i$. Let τ be the ordered partition $(1,1)$ of 2 and γ the coloring of $\mathcal{T}_\tau(\mathfrak{C})$ given by

$$\gamma(K)=\begin{cases}0, & \text{if } K \text{ is not an edge of } \mathfrak{G};\\ 1, & \text{otherwise.}\end{cases}$$

According to Lemma 3, there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ such that γ is constant on $\mathcal{T}_\tau(\mathfrak{D})$. If γ would be the function constant to 1, then $\mathfrak{G}$ would contain an infinite complete subgraph. Hence γ is constantly 0, which implies that $\mathfrak{D}$ is an induced subgraph of $\mathfrak{G}$. □

Theorem 2. *If, for every $n\in\omega$, $\mathfrak{G}\notin\mathcal{N}_n$, then, for every $\tau\in\omega$,*

$$\mathfrak{G}\leftrightarrow_{p(k)}(\aleph_0)^k_\tau.$$

Proof. If $\mathfrak{G}$ contains an infinite complete subgraph, the theorem follows from Theorem 1. If $\mathfrak{G}$ does not contain an infinite complete subgraph, then we deduce Theorem 2 from Lemmas 5 and 4. □

Lemma 6. *For every $1\leqslant n\in\omega$, a graph in $\mathcal{N}_n$ with infinite chromatic number contains an infinite induced path or it contains an induced stepgraph with infinite chromatic number.*

Proof. The lemma is obviously true for $n\leqslant 2$. We proceed by induction on n. Let $\mathfrak{G}\in\mathcal{N}_{n+1}$ be a K_{n+1}-free graph. If $\mathfrak{G}$ has infinite chromatic number then either $\mathfrak{G}$ contains a connected stepgraph with infinite chromatic number as induced subgraph or one of the connected components of $\mathfrak{G}$ has infinite chromatic number. Hence we may assume that $\mathfrak{G}$ is connected. We will either find an infinite path or a connected stepgraph with infinite chromatic number as induced subgraph of $\mathfrak{G}$, or we will be able to construct a sequence $((x_l, \mathfrak{G}_l; l \in \omega)$ such that, for each $l \in \omega$:

(i) $\mathfrak{G}$ restricted to $\{x_0, x_1, , , , . x_l\}$ is an induced path of $\mathfrak{G}$;

(ii) x_l is adjacent to x_{l+1};

(iii) $\mathfrak{G}_l \cup \{x_l\}$ is a connected infinite chromatic subgraph of $\mathfrak{G}$; and

(iv) for all $i < l$, x_i is not adjacent to any vertex in $\mathfrak{G}_l$.

Let x_0 be some vertex of $\mathfrak{G}$ and put $\mathfrak{G}_0 = \mathfrak{G}$. Assume that $l \in \omega$ and, for all $i < l$, $(x_i, \mathfrak{G}_i)$ has already been chosen. Let N be the set of those vertices of $\mathfrak{G}_l$ which are adjacent to x_l. The graph $\mathfrak{G}_l \mid N$ is in $\mathcal{N}_n$, for otherwise x_l would be contained in a K_{n+1}. Using induction, $\mathfrak{G}_l \mid N$ either contains an infinite path or a connected stepgraph with infinite chromatic number or has finite chromatic number. If it has finite chromatic number, the graph $\mathfrak{G}_l - N$ has infinite chromatic number. If $\mathfrak{G}_l - N$ does not contain a connected stepgraph with infinite chromatic number, then one of the connected components of $\mathcal{G}_l - N$ must have infinite chromatic number. Let $\mathfrak{H}$ be a connected component of $\mathfrak{G}_l - N$, which has infinite chromatic number. Because $\mathfrak{G}$ is connected, one of the vertices in N, say x_{l+1}, must be adjacent to some vertex in $\mathfrak{H}$. Let $\mathfrak{G}_{l+1}$ be the subgraph of $\mathfrak{G}_l$ induced by the vertices of $\mathfrak{H}$. It is now easy to check that the sequence $((x_i, \mathfrak{G}_i)$; $i \leqslant l+1)$ satisfies the four conditions listed above. □

THEOREM 3. *If for every $n \in \omega$, $\mathfrak{G} \notin \mathcal{N}_n$ but $\mathfrak{G}$ does not contain an infinite complete subgraph, then $\mathfrak{G} \not\leftrightarrow_{p(k)-1} (\aleph_0)^k_{p(k)}$.*

PROOF. Fix a bijection α from G to ω. If $\mathfrak{A}$ and $\mathfrak{B}$ are two induced subgraphs of $\mathfrak{G}$, we write $\mathfrak{A} < \mathfrak{B}$ if for every vertex $a \in A$ and vertex $b \in B$, $\alpha(a) < \alpha(b)$. Let S be a k-element subset of G. The set S is a *partitioned* subset of G if the connected components of $\mathfrak{G}|_S$ are totally ordered by $<$. If $P_1(S) < P_2(S) < \cdots < P_w(S)$ are the connected components of the partitioned subset S of G, then we associate with S the ordered partition $\tau = (|P_1(S)|, |P_2(S)|, \ldots, |P_w(S)|)$. If S is not a partitioned subset of G we associate arbitrarily some ordered partition of k with S. This provides a coloring γ of all k-element subsets of G with the ordered partitions of k.

Next we prove that if $\mathfrak{H}$ is an induced subgraph of $\mathfrak{G}$ which has infinite chromatic number, then the restriction of γ to $[H]^k$ is a surjection onto all ordered partitions of k. (This of course will imply the theorem.)

Assume that $\mathfrak{H}$ contains a connected stepgraph $\mathfrak{C}$ as an induced subgraph. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$ be an ordered partition of k. Choose an index i_1 such that $|P_{i_1}(\mathfrak{C})| > \tau_1$ and a subset $S_1 \subseteq P_{i_1}(\mathfrak{C})$ such that $|S_1| = \tau_1$ and $\mathfrak{C}|_{S_1}$ is connected. Assume that $S_1, S_2, \ldots, S_v$ for $v < w$ have already been chosen, so that, for all $1 \leqslant i \leqslant v$, $|S_i| = \tau_i$, $\mathfrak{C}|_{S_i}$ is connected and, for all $1 \leqslant i \leqslant v-1$, $S_i < S_{i+1}$. Because there are only finitely many numbers which are not larger than all of the numbers $\alpha(x)$ with $x \in \bigcup_{1 \leqslant i \leqslant v} S_i$, there is an index j such that

$$\bigcup_{1 \leqslant i \leqslant v} S_i < \bigcup_{j \leqslant i \in \omega} P_i(\mathfrak{C}) \qquad \text{and} \qquad |P_j(\mathfrak{C})| > \tau_{v+1}.$$

We choose $S_{v_1} \subseteq P_j(\mathfrak{C})$ such that $|S_{v+1}| = \tau_{v+1}$ and $\mathfrak{C}|_{S_{v+1}}$ is connected. This means that the sequence $(S_1, S_2, \ldots, S_v)$ can be extended, and hence there exists, for every ordered partition τ of k, a k-element subset $K \subseteq C$ with $\gamma(K) = \tau$. We conclude that if $\mathfrak{H}$ contains a stepgraph, then γ restricted to $[H]^k$ is onto, and hence we are done.

It now follows from Lemma 5 that we may assume that the clique sizes of $\mathfrak{H}$ are bounded and from Lemma 6 that we may assume that $\mathfrak{H}$ contains an infinite induced path $\mathfrak{M}$. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$ be an ordered partition of k. We wish to construct a partitioned subset K of W such that $\gamma(K) = \tau$ by successively constructing $P_1(K) < P_2(K)$ and so on. Choose for $P_1(K)$ any connected induced subgraph of $\mathfrak{M}$ containing τ_1 vertices. Assume that, for some $v < w$ and all $i \leqslant v$, the sets $P_i(K)$ have already been found. The function α takes on a maximal finite value m on the sets $P_i(K)$ and hence there are only finitely many vertices x in $\mathfrak{M}$ such that $\alpha(x) \leqslant m$. That implies that we can find a subset $P_{v+1}(K)$ having the required properties. □

3. The Results for Graphs with Bounded Clique Sizes

Here we have to use a result from finite Ramsey theory. First, we present some notation. The set of all induced subgraphs of a graph $\mathfrak{G}$ which are isomorphic to the graph $\mathfrak{K}$ is denoted by $\binom{\mathfrak{G}}{\mathfrak{K}}$.

$$\mathfrak{G} \rightarrowtail (\mathfrak{H})^{\mathfrak{K}}_{\tau}$$

means that, for every coloring $\gamma: \binom{\mathfrak{G}}{\mathfrak{K}} \to \tau$, there is an induced subgraph $\mathfrak{H}'$ isomorphic to $\mathfrak{H}$ of $\mathfrak{G}$ such that γ is constant on $\binom{\mathfrak{H}'}{\mathfrak{K}}$. An ordered graph $(\mathfrak{G}, \leqslant)$ is a graph together with a total order on G. Isomorphisms and embeddings of ordered graphs must also respect the order. The set of all induced subgraphs of an ordered graph $(\mathfrak{G}, \leqslant)$ which are isomorphic to the ordered graph $(\mathfrak{K}, \leqslant)$ is denoted by $\binom{(\mathfrak{G}, \leqslant)}{(\mathfrak{K}, \leqslant)}$. The expression

$$(\mathfrak{G}, \leqslant) \rightarrowtail (\mathfrak{H}, \leqslant)^{(\mathfrak{K}, \leqslant)}_{\tau}$$

means that, for every coloring $\gamma: \binom{(\mathfrak{G}, \leqslant)}{(\mathfrak{K}, \leqslant)} \to \tau$, there is an induced subgraph $(\mathfrak{H}', \leqslant)$ isomorphic to $(\mathscr{H}, \leqslant)$ of $(\mathfrak{G}, \leqslant)$ such that γ is constant on $\binom{(\mathfrak{H}', \leqslant)}{(\mathscr{H}, \leqslant)}$. The following theorem is due to Nešetřil and Rödl [7, 8] and independently, Abramson and Harrington (1).

Theorem. *Given $\tau \in \omega$ and finite ordered graphs $(\mathfrak{H}, \leqslant)$ and $(\mathfrak{K}, \leqslant)$ such that $\mathscr{H}, \mathfrak{K} \in \mathscr{N}_n$, there exists a finite ordered graph $(\mathfrak{G}, \leqslant)$, with $\mathfrak{G} \in \mathscr{N}_n$ such that*

$$(\mathfrak{G}, \leqslant) \rightarrowtail (\mathfrak{H}, \leqslant)^{(\mathfrak{K}, \leqslant)}_{\tau}.$$

Note that this theorem implies the following, as has been observed by the above authors.

Statement *. *If the group of automorphisms of $\mathfrak{K}$ is the full symmetric group on K, that is if $\mathfrak{K}$ is either complete or does not contain any edges and*

$$(\mathfrak{G}, \leqslant) \rightarrowtail (\mathfrak{H}, \leqslant)^{(\mathfrak{K}, \leqslant)}_{\tau},$$

then

$$\mathfrak{G} \rightarrowtail (\mathfrak{H})^{\mathfrak{K}}_{\tau}. \tag{3}$$

In particular, (3) *holds if $\mathfrak{K}$ is a single edge or consists of two non adjacent vertices. If $\mathfrak{H} \in \mathscr{N}_n$ we can also require that $\mathfrak{G} \in \mathscr{N}_n$.*

The symbol

$$\mathfrak{G} \rightarrowtail (\mathfrak{H})^2_{\kappa, v}$$

means that, for every $\gamma: \mathfrak{G} \to \kappa$ and for every $\delta: ([G]^2 - \mathfrak{G}) \to v$, there is an induced subgraph $\mathfrak{H}'$ of $\mathfrak{G}$ isomorphic to $\mathfrak{H}$ such that γ is constant on $\mathfrak{H}'$ and v is constant on $\overline{\mathfrak{H}'}$ (the edge set of the complement of $\mathfrak{H}'$). We need the following result:

$$\forall\, \mathscr{Z} \leqslant n < \omega\ \forall \tau < \omega\ \forall \mathfrak{H} (|\mathfrak{H}| < \omega \wedge \mathfrak{H} \in \mathscr{N}_n) \Rightarrow \exists \mathfrak{G} (\mathfrak{G} \in \mathscr{N}_n \wedge |\mathfrak{G}| < \omega \wedge \mathfrak{G} \rightarrowtail (\mathfrak{H})^2_{\tau, \tau}). \tag{4}$$

It is easy to see that (4) follows from statement $*$ by 'Ramseying twice': that is, by first finding a graph $\mathfrak{L} \in \mathcal{N}_n$ such that $\mathfrak{L} \rightarrowtail (\mathfrak{H})^{\mathfrak{N}}_{\tau}$, where $\mathfrak{N}$ is the graph on two vertices without an edge, and then by finding a graph $\mathfrak{G} \in \mathcal{N}_n$ such that $\mathfrak{G} \rightarrowtail (\mathfrak{L})^{\mathfrak{K}}_{\tau}$, where $\mathfrak{K}$ is the graph on two adjacent vertices. Clearly, $\mathfrak{G} \in \mathcal{N}_n$ and $\mathfrak{G} \rightarrowtail (\mathfrak{H})^2_{\tau,\tau}$. Another interesting way of proving (4) has recently been found by Shelah. In [6], Komjáth and Shelah prove a consistency result which was then generalized by Shelah at the Banff conference on 'Finite and Infinite Combinatorics in Sets and Logic', in April 1991 to: the following is consistent for every $\lambda \geqslant \mathcal{Z}$;

$$\forall \kappa\ \forall \nu\ \forall \mathfrak{H} \in \mathcal{N}_\lambda\ \exists \mathfrak{G} \rightarrowtail (\mathfrak{H})^2_{\kappa,\nu}. \tag{5}$$

Using compactness and absoluteness, one obtains (4) from (5) (Shelah).

Lemma 7. *If $\mathfrak{G}$ is a finite graph $\mathfrak{G} \leftrightarrow_s (sk)^2_{rl+\binom{l}{2}}$ and γ is a coloring of G with l colors, then there exists an $i < l$ such that $\mathfrak{G}|_{\gamma^{-1}(i)} \leftrightarrow_s (k)^2_\tau$.*

Proof. Otherwise,

$$\forall i < l, \qquad \mathfrak{G}|_{\gamma^{-1}(i)} \not\leftrightarrow_s (k)^2_\tau. \tag{6}$$

For $i < l$, fix a coloring δ_i witnessing (6). We may assume that different color sets are used for distinct values of i. We define a coloring δ of $[G]^2$ as follows. If, for $i < \tau$, $e \in \gamma^{-1}(i)$, then $\delta(s) = \delta_i(e)$. For $i \neq j$, $x \in \gamma^{-1}(i)$ and $y \in \gamma^{-1}(j)$, $\delta(\{x, y\}) = \{i, j\}$. If $T \subseteq G$ and $|\delta([T]^2)| \leqslant s$, then there are at most s different numbers i such that $T \cap \gamma_i^{-1} \neq \varnothing$. But then it follows from (6) that the chromatic number of $\mathfrak{G}|_T$ is less than sk. Hence δ is a witness to

$$\mathfrak{G} \not\leftrightarrow_s (sk)^2_{rl+\binom{l}{2}}$$

and we have arrived at a contradiction. $\square$

We will use Lemma 7 in the following form.

Corollary 1. *For every three integers $2 \leqslant s, k, l \in \omega$ there is a number $N = N(s, k, l)$ such that whenever $\mathfrak{G}$ is a graph with $\mathfrak{G} \leftrightarrow_s (N)^2_N$ and γ is a function from G into l, then, for some $i \in l$,*

$$\mathfrak{G}|_{\gamma^{-1}(i)} \leftrightarrow_s (k)^2_k.$$

Let $(C_i; i \in \omega)$ be a sequence of pairwise disjoint finite sets. An *s-chromatic stepgraph* $\mathfrak{C} = (C, E)$ *with partition* $C_0, C_1, C_2, \ldots$ is a graph such that $C = \bigcup_{i \in \omega} C_i$ and, for all $i \in \omega$ $(\mathfrak{C}|_{C_i} \leftrightarrow_s (i)^2_i)$. We define the *partition type* and other notions as for ordinary stepgraphs. A *refinement* of an s-chromatic stepgraph $\mathfrak{C}$ is an induced subgraph of $\mathfrak{C}$ which is also an s-chromatic stepgraph. We obtain the following immediately from Corollary 1.

Lemma 8. *If γ is a coloring of the vertices of an s-chromatic stepgraph $\mathfrak{C}$ with finitely many colors, then there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ such that γ is constant on the vertices of $\mathfrak{D}$.*

Lemma 9. *Let $\mathfrak{C} = (C, E)$ be an s-chromatic stepgraph with partition $(C_0, C_1, C_2, \ldots)$, τ the ordered partition $(1, 1)$ of 2 and $\gamma : \mathcal{T}_\tau(\mathfrak{C}) \to \tau$ a coloring of the 2-element subsets of C which have partition type τ with $\tau \geqslant 1$ colors. Then, for every*

$n \in \omega$, there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ with $P_i(\mathfrak{D}) = C_i$ for $i < n$ and such that the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform up to n.

Proof. We define the number c to be

$$c = |\tau^{\bigcup_{i<n} C_i}|.$$

Note that $|\tau^{\bigcup_{i<n} C_i}|$ is the number of colorings of the set of all elements of $S = \bigcup_{i<n} C_i$, with τ colors. Let $j \geq n$ and $x \in C_j$. Note that if $y(\{x, y\}$. Then $\gamma_x \in \tau^s$. Now consider the association of x with γ_x to be coloring Γ of $\bigcup_{i \geq n} C_i$ with c colors.

There exists a sequence t_i such that

$$\forall i < n,\ t_i = i \wedge \forall i \geq n, \qquad \mathfrak{C}|_{C_{t_i}} \leftrightarrow_s (N(s, i, c))^2_{N(s,i,c)}.$$

By the definition of the number $N(s, i, c)$, for each $i \geq n$, there is a subset $B_i \subseteq C_{t_i}$ such that Γ is constant on $[B_i]^k$. Put $B_i = C_i$ for $i < n$ and $\mathfrak{B} = \mathfrak{C}|_{\bigcup_{i\in\omega} B_i}$. Because the coloring Γ is finite, there is a refinement $\mathfrak{D}$ of $\mathfrak{B}$ such that Γ is constant on $\bigcup_{i \geq n} P_i(\mathfrak{D})$. Clearly, $\mathfrak{D}$ is the desired refinement of $\mathfrak{C}$ for which the restriction of γ from $\mathcal{T}_\tau(\mathfrak{C})$ to $\mathcal{T}_\tau(\mathfrak{D})$ is uniform up to n. □

Lemma 10. *If $\mathfrak{C}$ is an s-chromatic stepgraph and $\gamma : \mathcal{T}_{(1,1)}(\mathfrak{C}) \to \tau$ is a coloring of the pairs that connect different parts of $\mathfrak{C}$ with τ colors, then there is a refinement $\mathfrak{D}$ of $\mathfrak{C}$ such that γ is constant on $\mathcal{T}_{(1,1)}(\mathfrak{D})$.*

Proof. Using Lemma 9 repeatedly, we can construct as in the proof of Lemma 2 a refinement $\mathfrak{B}$ of $\mathfrak{C}$ such that γ is n-uniform on $\mathfrak{B}$ for every $n \in \omega$, which of course means that γ is uniform on $\mathfrak{B}$. We obtain that there exists a function $\rho : B \to \tau$ such that

$$\forall i \in \omega\ \forall x \in P_i\ \forall i < j \in \omega\ \forall y \in P_j(\mathfrak{B}), \qquad \gamma(\{x, y\}) = \rho(x).$$

According to Lemma 9, there is a restriction $\mathfrak{D}$ of $\mathfrak{B}$ on which the function ρ is constant. Clearly, then, γ is constant on $\mathcal{T}_{(1,1)}(\mathfrak{D})$. □

Lemma 11. *If $\mathfrak{C}$ is an s-chromatic stepgraph, then, for all $2 \leq \tau \in \omega$, $\mathfrak{C} \leftrightarrow_{s+1} (\aleph_0)^2_\tau$.*

Proof. If $\gamma : [C]^2 \to \tau$ is a coloring of the two-element subsets of C, there exists, according to Lemma 10, a refinement $\mathfrak{G}$ of $\mathfrak{C}$ such that that γ is constant on $\mathcal{T}_{(1,1)}(\mathfrak{G})$. Because $\mathfrak{C}|_{C_i} \leftrightarrow_s (i)^2_i$, there is an infinite chromatic subgraph $\mathfrak{S}$ of $\mathfrak{G}$ such that $|\gamma([S]^2)| \leq s + 1$. Hence, for every $1 \leq \tau \in \omega$, $\mathfrak{G} \leftrightarrow_{s+1} (\aleph_0)^2_\tau$. □

Theorem 4. *There exists a triangle-free graph $\mathfrak{G}$ such that, for each $1 \leq \tau \in \omega$, $\mathfrak{G} \leftrightarrow_{\mathscr{A}} (\aleph_0)^2_\tau$.*

Proof. Let $(\mathfrak{H}_i; i \in \omega)$ be a family of finite graphs such that the chromatic number of $\mathfrak{H}_i$ is larger than i. We use (4) to obtain a family $(\mathfrak{C}_i; i \in \omega)$ of finite graphs such that $\mathfrak{C}_i \rightarrowtail (\mathfrak{H}_i)^2_{i,i}$. Note that, for each $i \in \omega$, $\mathfrak{C}_i \leftrightarrow_2 (i)^2_i$ and hence it is obvious that the graph $\mathfrak{C}$ with $P_i(\mathfrak{C}) = C_i$ is a 2-chromatic stepgraph. It now follows from Lemma 11 that, for every $1 \leq \tau \in \omega$, $\mathfrak{G} \leftrightarrow_{3} (\aleph_0)^2_\tau$. □

Theorem 5. *There exists a triangle-free graph $\mathfrak{G}$ such that, for every $2 \leq \tau \in \omega$, $\mathfrak{G} \not\leftrightarrow_{r-1} (\aleph_0)^2_\tau$.*

Proof. Let $\mathfrak{G}$ be a stepgraph such that, for every $i \in \omega$, the part $\mathfrak{G}_i$ has chromatic number at least i and girth at least i. Such graphs exist, by [8]. Denote by $d(x, y)$ the distance between the vertices x and y in $\mathfrak{G}$. Color the elements of $[G]^2$ with τ colors according to distance modulo τ; that is,

$$\forall x \neq y \wedge x, y \in G, \qquad \gamma(\{x, y\}) = d(x, y) \quad \text{modulo} \quad \tau.$$

Every infinite chromatic subgraph of $\mathfrak{G}$ must contain subgraphs of large chromatic numbers of infinitely many of the graphs $\mathfrak{G}_i$, and hence pairs in all of the τ colors. □

THEOREM 6. *For $3 \leq n \in \omega$ and $3 \leq \tau \in \omega$, $\mathfrak{U}_n \not\leftrightarrow_2 (\aleph_0)^2_\tau$.*

PROOF. Let $\mathfrak{U}_n$ be the homogeneous universal K_n-free graph and suppose that $<$ well orders $|U_n|$ in order type ω. We fix a coloring γ of $[U_n]^2$ as follows: edges are colored black; if $\{a, b\} \notin \mathfrak{U}_n$ and $a < b$, then $\{a, b\}$ is colored red just in case there is an $x < a$ with $\{x, b\} \in \mathfrak{U}_n$; otherwise $\{a, b\}$ is colored blue.

We prove that any infinite chromatic subgraph $\mathfrak{H}$ of $\mathfrak{G}$ must use all three colors. Note that this argument also proves that the complement of $\mathfrak{U}_m$ is not weakly edge indivisible.

Specifically, we prove by induction on $m \leq n$ that if $\mathfrak{H} \subseteq \mathfrak{U}_n$ is homogeneous in two colors and K_m-free, then $\chi(\mathfrak{H})$ is finite. It is clear that unless one of the colors used is black, the chromatic number of a homogeneous subgraph is one, as the graph is edge-free. Let $\mathfrak{H}$ be K_m-free and homogeneous in two colors. If $m < 2$, $\mathfrak{H}$ is edge-free, and colorable with one color. We are led to two cases.

Case 1: *$\mathfrak{H}$ is homogeneous with black edges and red non-edges.* Let h_0 be the least element of $\mathfrak{H}$. Partition $\mathfrak{H}$ into finitely many classes: $X_{-1} = \{h_0\}$ and, for Let h_0 be the least element of $\mathfrak{H}$. Partition $\mathfrak{H}$ into finitely many classes: $X_{-1} = \{h_0\}$ and, for $0 \leq i \leq h_0$, $X_i = \{h \in \mathfrak{H}$ such that $i = \min\{x \in \mathfrak{U}_n : \{x, h\} \in \mathfrak{U}_n\}\}$. This is a partition of $\mathfrak{H}$, since either $h_0 = h$, $h_0 < h \in \{h_0, h\} \in \mathfrak{U}_n$ or $h_0 < h$ and $\gamma(\{h_0, h\}) = \text{red}$, so $\exists i < h_0(\{i, h\} \in \mathfrak{G}))$. Now each X_i induces a subgraph of $\mathfrak{U}_n$ which is K_{n-1}-free, and homogeneous in black and red. By the induction hypothesis, $\chi(X_i) = \eta_i < \infty$, so $\chi(\mathfrak{H}) \leq \sum_{i=-1}^{h_0} \eta_i < \infty$, as required.

Case 2: *$\mathfrak{H}$ is homogeneous in black and blue.* We argue that, in fact, $\chi(\mathfrak{H}) < m$. Otherwise there is a finite $\mathfrak{K} \subseteq \mathfrak{H}$ which unduces a subgraph of chromatic number of at least m. It must be the case that some member of $\mathfrak{K}$, h_0 say, has at least $m - 1$ neighbors $h \in \mathfrak{K}$ with $h_0 < h$; otherwise, it has a coloring number of at most $m - 1$. So let $h_0, h_1, \ldots, h_{m-1} \in \mathfrak{H}$ with $\{h_0, h_i\} \in \mathfrak{U}_n$ for $0 < i \leq m - 1$ and with $h_0 < h_1 < \cdots < h_{m-1}$. Observe that, for $0 < i < j \leq m - 1$, we must have $\{h_i, h_j\} \in \mathfrak{U}_n$; for otherwise the fact that $\{h_0, h_j\} \in \mathfrak{U}_n$ would entail that the non-edge $\{h_i, h_j\}$ is colored red. But then $\{h_0, h_1, \ldots, h_{m-1}\}$ induces a complete graph K_m in $\mathfrak{H}$, contrary to assumption.

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